Application of Differential Transform Method with Adomian Polynomial for solving RLC Circuits Problems and Higher Order differential equations

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Abstract

The differential transform method (DTM) for solving linear and nonlinear higher order differential equations especially which arising in the field of electric circuit problems is used. In almost cases, DTM gives the series of solutions which can be easily converted to exact ones. In nonlinear problems it is useful to use the Adomian polynomial to overheads the nonlinear terms. Some illustrative examples are presented with some RLC circuit problems to show the simplicity and accuracy of the used method compared with other numerical methods. All calculations are made with the help of computer algebra software Mathematica 11.2.

Keywords : Differential transform method, Higher order, Initial-boundary value problems, RLC circuit problems, Adomian polynomial.

1. Introduction

Many problems in engineering and physics and other fields reduce a differential equations or partial differential equations. Some of these problems are linear the other are nonlinear. Finding the exact solution in almost applications is hard and complicated, so the numerical solution in this cases is effective tool. The differential transform method (DTM) is a semi-analytical and numerical method for solving a wide variety of differential equations and usually gets the solution in a series form.

Differential transform method (DTM) was firstly introduced by Zahou [1] in 1986 to solve linear and nonlinear initial value problems in electric circuit analysis,

This method introduces a semi-analytic solution for differential equations, in polynomial form, it differs from the regular high order Taylor series technique, which necessitates the symbolic calculation of the necessary derivatives for the data functions. Computationally, for large orders the Taylor series takes a long time. But, the DTM is a method for iteratively obtaining analytical Taylor series solutions to differential equations, with simple calculations. Moreover, DTM can be applied

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to all the different types of differential and integral equations which can be summarized as:

Eigenvalue problems in [2] by Chen and Ho (1996), in [3] Hassan (2007) he applied the DTM on the one-dimensional planar Bratu problem.

Initial value problems in [4] Jang and Chen (2000) used the DTM to approximate the solutions of linear and nonlinear initial value problems, in [5] Hassan (2008) he made a comparison between DTM and ADM for partial differential equations.

Partial differential equations in [6] Kumari (2015) use the Laplace DTM to solve wave equations and wave-like equations.

Integral and integro-differential equations [7], in [8] Kajani and Shehni (2011) used the DTM to solve nonlinear Volterra integro-differential equations.

System of differential equations in [5] Hassan (2008) solved system of differential equations.

Delay differential equations in [9] Karakoç and Bereketoglu (2009)and in [10] Alaa (2016) used the DTM to solve Delay differential equations.

Engineering problems one of the most important application in our life [11], in this work we concentrate on RLC problems which are fundamental in many applications involving low and high frequency signals [12], RLC circuits can be described by a second-order differential equation for the analysis of the circuit.

In this paper we extend the application of differential transform technique to create a semi-analytical solution of the higher order nonlinear differential equations of electric circuits.

This paper is organized as follows: In section 2, the differential transform method is described with Adomian polynomial technique [13].

In section 3,the method is implemented to some experiments and RLC circuit

problems [12], finally the conclusion is given in section 4.

2.   METHOD OF SOLUTION

2.1 The Differential Transform procedure

The differential transform of the kth derivative offunction $y(x)$ is defined as following

In order to illustrate the advantages and the accuracy of the DTM for solving the linear and nonlinear higher order differential equations and

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \qquad (1)$$

where Y(k), is the transformed function and y(x) is the original function. Differential inverse transform of Y(k) is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \approx y_N(x) = \sum_{k=0}^{N} Y(k)x^k \qquad (2)$$

By substituting equation (1) in (2) we get

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{d^k y(x)}{dx^k}\bigg|_{x=0} \qquad (3)$$

it indicates that the differential transform concept is derived by Taylor series expansion. In the previous definition we consider the case when $x = 0$, but it is true for any fixed real number $x = x_0$. The main theorems that can be derived from equations (1) and (2) can be summarized in table (1) see [10].

2.2 Adomian Polynomials

Solving the higher order boundary value problems is hard and difficult especially for that which contains nonlinear terms, so in this case we

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left[\sum_{m=0}^{\infty} \lambda^m y_m\right]_{\lambda=0}$$

use DTM modified by Adomian polynomials to facilitate the calculations and make it more simple, see [10], [13]. Nonlinear term $Ny(x)$ is decomposed as a sum of polynomials denoted by A_n. i.e.

$$Ny(x) = \sum_{n=0}^{\infty} A_{n,} \qquad (4)$$

Where

$$A_0 = N(y_0), A_1 = N'(y_0)y_1, A_2 = N'(y_0)y_2 + \frac{1}{2!}N''(y_0)y_1^2,$$

$$A_3 = N'(y_0)y_3 + \frac{2}{2!}N''(y_0)y_1 y_2 + \frac{1}{3!}N'''(y_0)y_1^3, \cdots \qquad (5)$$

3.   APPLICATIONS AND NUMERICAL RESULTS

RLC circuit problems, some illustrative example are introduced.

<div align="center">**Table 1** : The fundamental operations of DTM</div>

Original functions	Transformed functions
$y(x) = u(x) \pm m(x)$	$Y(k) = U(k) \pm M(k)$
$y(x) = \alpha m(x)$	$Y(k) = \alpha M(k)$
$y(x) = \dfrac{du(x)}{dx}$	$Y(k) = (k+1)U(k+1)$
$y(x) = \dfrac{d^2 u(x)}{dx^2}$	$Y(k) = (k+1)(k+2)U(k+2)$
$y(x) = \dfrac{d^v u(x)}{dx^v}$	$Y(k) = (k+1)(k+2)\mathrm{K}(k+n)U(k+n)$
$y(x) = x^m$	$Y(k) = \delta(k-m) = \begin{cases}1, & k=m \\ 0, & k \neq m\end{cases}$
$y(x) = g(x)h(x)$	$\displaystyle\sum_{m=0}^{k} H(m)G(k-m)$
$y(x) = e^{\lambda x}$	$Y(k) = \dfrac{\lambda^k}{k!}$
$y(x) = (1+x)^m$	$Y(k) = \dfrac{m(m-1)\ldots(m-k+1)}{k!}$
$f(y) = y^m$	$F(k) = \begin{cases} Y^m(0) & \text{if } k=0, \\ \dfrac{1}{Y(0)}\displaystyle\sum_{r=1}^{k} \dfrac{(m+1)r-k}{k} Y(r)F(k-r) & \text{if } k \geq 1. \end{cases}$
$f(y) = e^{ay}$	$F(k) = \begin{cases} e^{aY(0)} & \text{if } k=0 \\ a\displaystyle\sum_{r=0}^{k-1} \dfrac{r+1}{k} Y(r+1)F(k-r-1) & \text{if } k \geq 1 \end{cases}$
$f(y) = sin(ay)$	$F(k) = \begin{cases} \sin(aY(0)) & \text{if } k=0 \\ a\displaystyle\sum_{r=0}^{k-1} \dfrac{k-r}{k} G(r)Y(k-r) & \text{if } k \geq 1 \end{cases}$

Example (3.1): Consider the 10th order nonlinear boundary value problem[14]:

$$y^{(10)}(x) = e^{-x}y^2(x), \qquad 0 \leq x \leq 1 \tag{6}$$

subject to the following conditions:

$$\begin{cases} y^{(\sigma)}(0) = 1 & \sigma = 0,2,4,6,8, \\ y^{(\sigma)}(1) = e & \sigma = 0,2,4,6,8. \end{cases} \tag{7}$$

Which has the exact solution

$$y(x) = e^x.$$

Solution. Apply the DTM on Equation (6), we have

$$Y(k+10) = \frac{\sum_{r=0}^{k} \frac{(-1)^r}{r!} A[k-r]}{(k+10)(k+9)(k+8)(k+7)(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)} \tag{8}$$

Where $A[k-r]$ is the adomian polynomials of $y^2(x)$.

$A_0 = Y^2(0), A_1 = 2Y(0)Y(1), A_2$
$\qquad = Y^2(1) + 2Y(0)Y(2),$
$A_3 = 2Y(1)Y(2) + 2Y(0)Y(3),..$ 　　　　(**9**)

From the condition (7) we get

$Y[0] = 1; Y[1] = a; Y[2] = 0.5; Y[3] = b; Y[4] = \dfrac{1}{4!};$

$Y[5] = c; Y[6] = \dfrac{1}{6!}; Y[7] = d; Y[8] = \dfrac{1}{8!}; Y[9] = f;$

Using recurrence relation (8) and (9) for
$k = 0,1, \dots ,4$ we get the following
$Y[10] = \dfrac{1}{10!}, \ Y[11] = \dfrac{-1+2a}{11!}, \cdots$
We can write the solution as

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k$$

$y(x) = 1 + ax + 0.5x^2 + bx^3 + \dfrac{x^4}{24} + cx^5 + \dfrac{x^6}{720} + dx^7$
$\qquad + \dfrac{x^8}{40320} + fx^9 + \dfrac{x^{10}}{3628800}$
$\qquad + \dfrac{(-1+2a)x^{11}}{399168} + 4.17535139$
$\qquad \times 10^{-9}(1.5 - 2.aa^2)x^{12} + (-1.124133$
$\qquad \times 10^{-9} + 1.92708526 \times 10^{-9}a$
$\qquad - 9.63542630 \times 10^{-10}a^2 + 1.92708526$
$\qquad \times 10^{-9}b)x^{13} + (2.40885657 \times 10^{-10}$
$\qquad + 1.37648947 \times 10^{-10}a^2$
$\qquad + a(-3.67063859 \times 10^{-10}$
$\qquad + 5.50595788 \times 10^{-10}b) - 5.50595788$
$\qquad \times 10^{-10}b)x^{14}$

Using the boundary condition (7) we can get
a, b, c, d, f
$\quad a = 1 \ \& \ b = 0.166666598 \& c = 0.00833336$
$\qquad \& \ d = 0.0001984 \& \ f = 0.00000275$

then $\therefore y(x) = 1 + x + 0.5x^2 + 0.166666598x^3$
$\qquad + \dfrac{x^4}{24} \qquad + 0.00833336x^5$
$\qquad + \dfrac{x^6}{720} + 0.0001984x^7 + \dfrac{x^8}{40320}$
$\qquad + 0.00000275x^9 + \dfrac{x^{10}}{3628800}$
$\qquad + 2.505211 \times 10^{-8}x^{11}$
$\qquad + 2.08767569 \times 10^{-9}x^{12}$
$\qquad + 1.6059034 \times 10^{-10}x^{13}$
$\qquad + 1.14707455 \times 10^{-11}x^{14}$

Table 2: Comparison of DTM solution with the exact solution and wavelet solution for Example (3.1)

x	Approximate	Exact	Absolute Error	Wavelet Absolute Error	[15]
0.0	1.000000000000000	1.000000000000000	0.00e-00	0.00e-00	0.00e-00
0.1	1.1051709221823827	1.1051709180756477	4.10e-09	7.08e-06	1.25e-06
0.2	1.2214027659745268	1.2214027581601699	7.81e-09	1.34e-05	8.69e-06
0.3	1.349858818337785	1.3498588075760032	1.07e-08	1.85e-05	2.14e-05
0.4	1.4918247103018856	1.4918246976412703	1.26e-08	2.18e-05	4.35e-06
0.5	1.6487212840234176	1.6487212707001282	1.33e-08	2.29e-05	3.96e-05
0.6	1.8221188130725088	1.822118800390509	1.26e-08	2.17e-05	5.40e-05
0.7	2.013752718266886	2.0137527074704766	1.07e-08	1.84e-05	6.79e-05
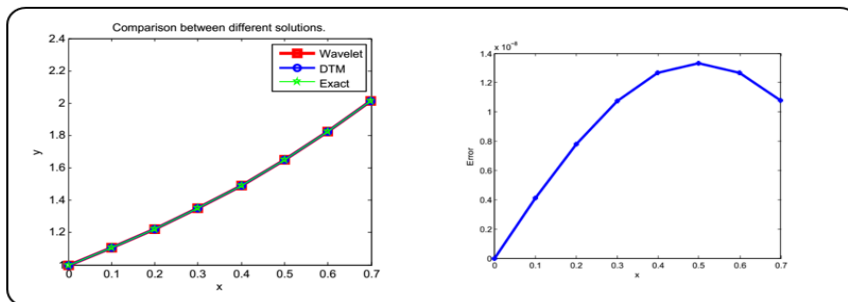
Figure (1) : Comparison between different methods of Solution for Example (3.1)

Example (3.2): Consider the following fourth order BVP [11]:

$$y^{(4)}(x) = y(x) + y''(x) + e^x(x-3), \qquad 0 < x < 1, (10)$$

subject to the boundary conditions

$$y(0) = 1, y'(0) = 0, y(1) = 0, y'(1) = -e. \quad (11)$$

The exact solution for this problem is
$$y(x) = (1-x)e^x. \qquad (12)$$

Solution. Apply the DTM on Equation (10), we have

$$Y(k+4)$$
$$= \frac{Y[k] + (k+1)(k+2)Y[k+2] + \sum_{r=0}^{k} \delta(r-1)\frac{1}{(k-r)!} - \frac{3}{(k)!}}{(k+1)(k+2)(k+3)(k+4)},$$
$$(13)$$

From the condition (11) we get
$$Y[0] = 1; Y[1] = 0; Y[2] = a; Y[3] = b;$$
Using recurrence relation (13) for $k = 0,1,\dots,7$ we get the following

$$Y[4] = \frac{1}{12}(-1+a) \dots \dots \dots \dots \dots Y[11]$$
$$= \frac{b}{1330560} \qquad (14)$$

We can write the solution as
$$y(x) = 1 + ax^2 + bx^3 + \frac{1}{12}(-1+a)x^4 + \frac{1}{60}(-1 + 3b)x^5 + \frac{1}{720}(-3+4a)x^6 + \frac{(-1+6b)x^7}{2520} + \frac{(-2+3a)x^8}{20160} + \frac{(-1+9b)x^9}{181440} + \frac{(-2+5a)x^{10}}{1814400} + \frac{bx^{11}}{1330560}$$

Using the boundary condition (11) we can get a, b
$$a = -0.49999977, b = -0.333333587$$
$$y(x) = 1 - 0.49999977x^2 - 0.333333587x^3 - 0.12499998x^4 - 0.033333346x^5 - 0.00694444x^6 - 0.001190476x^7 - 0.000173611x^8 - 0.00002204x^9 - 0.00000248x^{10} - 2.50521274 \times 10^{-7}x^{11}$$

Table 3 : Comparison of approximate and exact of Example (3.2)

x	Approximate	Exact	DTM Absolute Error	VIM Absolute Error
0.0	1.0000000000	1.0000000000	0.00e-00	0.00e-00
0.1	0.9946538282	0.9946538262	1.97e-09	7.08e-06
0.2	0.9771222134	0.9771222064	6.88e-09	1.34e-05
0.3	0.944901178573	0.9449011656	1.32e-08	1.85e-05
0.4	0.8950948382513	0.8950948188	1.96e-08	2.18e-05
0.5	0.82436065997	0.8243606355	2.46e-08	2.29e-05
0.6	0.728847546856	0.7288475200	2.66e-08	2.17e-05
0.7	0.604125836844	0.6041258121	2.46e-08	1.84e-05
0.8	0.44510820332	0.4451081856	1.76e-08	1.84e-05
0.9	0.2459603181764	0.2459603111	7.06e-09	1.84e-06
1.0	0.0000000000	0.0000000000	0.00e-00	00000

Example (3.3): Consider the nonlinear initial value problem

$$y'' = -\frac{g}{2}\left[3\frac{y}{R} - \left(\frac{y}{R}\right)^3\right], y(0) = \sqrt{6}R, y'(0) = 0, \qquad (15)$$

The exact solution for this problem is

$$y(x) = \sqrt{6}R * \text{Sec}\left[\sqrt{\frac{3g}{2R}} * x\right]. \qquad (16)$$

Solution. Apply the DTM on Equation (15), we have

$$Y(k+2)$$
$$= \frac{-\frac{3g*Y[k]}{2R} + \frac{g*A[k]}{2R^3}}{(k+1)(k+2)} \qquad (17)$$

Where $A[k]$ is the Adomian polynomials of $y^3(x)$.

$$A_0 = Y^3(0), A_1 = 3Y(0)^2Y(1), A_2$$
$$= 3Y(0)^2Y(2) + 3Y(0)Y^2(1),$$
$$A_3 = 3Y(0)^2Y(3) + 6Y(0)Y(1)Y(2) + Y(1)^3,$$
$$(18)$$

From the condition (15) we get
$$Y[0] = \sqrt{6}R; Y[1] = 0,$$
Using recurrence relation (8) and (9) for $k = 0,1,\dots$ we get the following

$$Y[2] = \frac{3}{2}\sqrt{\frac{3}{2}}g, Y[3] = 0, Y[4] = \frac{15\sqrt{\frac{3}{2}}g^2}{16R}, Y[5] = 0, ..$$

$$(19)$$

We can write the solution as

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k = \sqrt{6}R + \frac{3}{2}\sqrt{\frac{3}{2}}gx^2 + \frac{15\sqrt{\frac{3}{2}}g^2}{16R}x^4 + \cdots \quad (20)$$

which is the Taylor series expansion of the exact solution
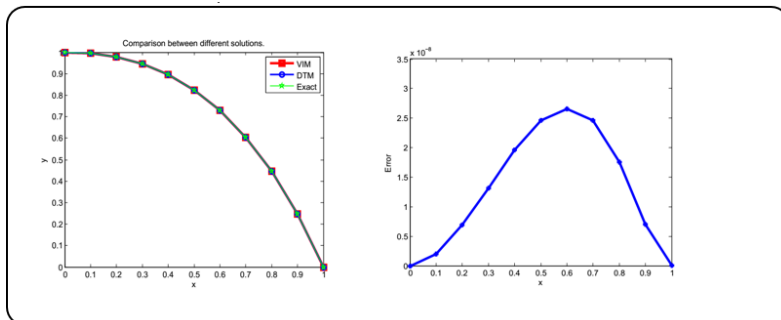
$$y(x) = \sqrt{6}R * Sec[\sqrt{\frac{3g}{2R}} * x].$$

Figure (2) : Comparison between different methods of Solution for Example (3.2)
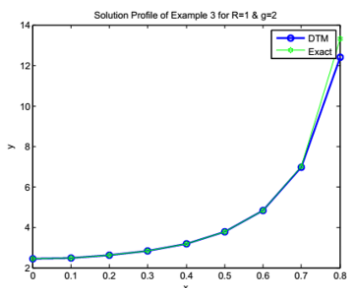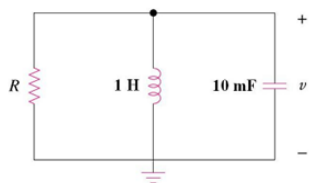

Figure (3) : comparison between DTM solution and exact solution of Example (3.3)

Application of DTM on Second-Order RLC Circuit

In this section, DTM method are applied on parallel RLC circuit (critically damped) [12] :

Find $v(t)$ for $t > 0$.

$$v(0) = 5V, i(0) = 0, R = 5\,\Omega, L = 1\,H, C = 10\,mF. \quad (21)$$

The exact solution for this problem is

$$v(t) = (5 - 50t)e^{-10t}. \quad (22)$$

Solution.

Initial conditions:

$$v(0) = 5, v'(0) = -\frac{v(0)+Ri(0)}{RC} = -100,$$

Applying KCL gives $\quad \frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt} = 0$ which gives

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0 \quad (23)$$

Apply the DTM on Equation (23), we have

$$V(K+2) = \frac{-\frac{V(K)}{LC} - \frac{(K+1)V(K+1)}{RC}}{(K+1)(K+2)}, \quad (24)$$

From the condition (21) we get

$$V[0] = 5; V[1] = -100,$$

Using recurrence relation (23) for $k = 0,1, ...$ we get the following

$$V[2] = 750, V[3] = \frac{10000}{3}, V[4] = \frac{31250}{3}, V[5]$$
$$= -25000, .. \quad (25)$$

We can write the solution as

$$v(t) = \sum_{k=0}^{\infty} V(k)t^k$$

$$v(t) = 5 - 100t + 750t^2 - \frac{10000}{3}t^3 + \frac{31250}{3}t^4 - 25000t^5 + \frac{437500}{9}t^6 + \cdots$$

which is the Taylor series expansion of the exact solution

$$v(t) = (5 - 50t)e^{-10t}.$$

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4. CONCLUSION

The main objective is to introduce Differential transform method (DTM) for solving different types of problems especially higher order linear and nonlinear differential equations with initial or boundary conditions and electric RLC circuit. For nonlinear problems the Adomian polynomial technique is used to compute the nonlinear terms. Some illustrative examples are introduced. These examples shows that the DTM is more accurate and simple comparing with other numerical methods such as variational iteration method [VIM], Wavelet approach method and Quintic B-spline collocation method.

5. ACKNOWLEDGEMENT

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